Corrected variant of the problem.
https://www.linkedin.com/feed/update/urn:li:activity:6636691762322767872
In any triangle with usual notations, prove that

$$(a+2r)(b+2r)(c+2r) \geq 2R^3(\sqrt{3}+5).$$

Solution by Arkady Alt, **San Jose** ,**California**, **USA**.
We will prove that in fact holds double inequality

(**1**) $\quad 16r^3(\sqrt{3}+5) \leq (a+2r)(b+2r)(c+2r) \leq 2R^3(\sqrt{3}+5).$

or, noting that $2(\sqrt{3}+5) = \left(\sqrt{3}+1\right)^3$ we rewrite (**1**) equivalently as

(**2**) $\quad 2r\left(\sqrt{3}+1\right) \leq \sqrt[3]{(a+2r)(b+2r)(c+2r)} \leq R\left(\sqrt{3}+1\right).$

1. By AM-GM Inequality we have

$$\sqrt[3]{(a+2r)(b+2r)(c+2r)} \leq \frac{1}{3}\sum(a+2r) = \frac{a+b+c+6r}{3}$$

and since $2r \leq R$ (Euler's Inequality) and $a+b+c \leq 3\sqrt{3}R$ ($a+b+c \leq \sqrt{3(a^2+b^2+c^2)}$ and $a^2+b^2+c^2 \leq 9R^2$) we obtain $\sqrt[3]{(a+2r)(b+2r)(c+2r)} \leq R\left(\sqrt{3}+1\right)$.

2. By replacing in Huygens Inequality $(x+1)(y+1)(z+1) \geq \left((xyz)^{1/3}+1\right)^3$ (x,y,z) with $\left(\frac{a}{2r},\frac{b}{2r},\frac{b}{2r}\right)$ we obtain

$$\left(\frac{a}{2r}+1\right)\left(\frac{b}{2r}+1\right)\left(\frac{b}{2r}+1\right) \geq \left(\sqrt[3]{\frac{a}{2r}\cdot\frac{b}{2r}\cdot\frac{b}{2r}}+1\right)^3 \Leftrightarrow$$

$$\sqrt[3]{(a+2r)(b+2r)(c+2r)} \geq \sqrt[3]{abc}+2r.$$

Since $(abc)^{1/3}+2r \geq 2r\left(\sqrt{3}+1\right) \Leftrightarrow (abc)^{1/3} \geq 2r\sqrt{3} \Leftrightarrow abc > 24r^3\sqrt{3} \Leftrightarrow$ $4Rrs > 24r^3\sqrt{3} \Leftrightarrow Rs > 6r^2\sqrt{3}$, where latter inequality holds because $R \geq 2r$ and $s \geq 3\sqrt{3}r$ then $(a+2r)(b+2r)(c+2r) \geq \left(2r\left(\sqrt{3}+1\right)\right)^3$